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STABILITY OF NEWTON BOUNDARIES OF A FAMILY OF REAL ANALYTIC SINGULARITIES

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ABSTRACT. Let $f_t(x, y)$ be a real analytic t-parameter family of real analytic functions defined in a neighborhood of the origin in $\mathbb{R}^2$. Suppose that $f_t(x, y)$ admits a blow analytic trivilaization along the parameter t (see the definition in §1 of this paper). Under this condition, we prove that there is a real analytic t-parameter family $\sigma_t(x, y)$ with $\sigma_0(x, y) = (x, y)$ and $\sigma_t(0, 0) = (0, 0)$ of local coordinates in which the Newton boundaries of $f_t(x, y)$ are stable. This fact claims that the blow analytic equivalence among real analytic singularities is a fruitful relationship since the Newton boundaries of singularities contains a lot of informations on them.

In an equisingular problem it is important to determine which equivalence among singularities is the best. It should be as strong as possible if the number of equivalence classes is kept in the admissible range. If we can find an appropriate equivalence relation, we will have a fruitful equisingular problem. In the theory of complex analytic singularities it is well known that the topological equivalence is just so.

Oka [7] shows the following: *If a complex analytic t-parameter family* $f_t(x, y)$ ($|t| \ll 1$) *of complex analytic functions in a neighborhood of the origin has a constant Milnor number* μ *at the origin and the Newton boundary of* f_0 *intersects the* x, y*-axes at two points, then there exists a complex analytic t-parameter family* $\sigma_t(x, y)$ *of local coordinates with* $\sigma_t(0, 0) = (0, 0)$ *and* $\sigma_0(x, y) = (x, y)$ *such that the Newton boundaries of* f_t *associated with the local coordinate* $\sigma_t(x, y)$ *are stable.* Since μ-constancy and topological constancy of $f_t^{-1}(0)$ are equivalent (see [6]), this result also means that the Newton boundaries of a topologically constant family $f_t(x, y)$ are stable in some complex analytic family $\sigma_t(x, y)$ of local coordinates. It is well known that the Newton boundaries of singularities have a lot of information on them. Hence the result claims that the topological equivalence among complex analytic singularities is strong enough and it is natural that the equisingular problem with respect to this equivalence relation is fruitful.

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Now we ask whether Oka's aforementioned result is correct for a real analytic family of real analytic functions with constant topological type. The answer is not affirmative. A counterexample is given as follows.

Example. Let $f_t(x, y)$ be the real analytic t-parameter family of real analytic functions in a neighborhood of the origin, given by the formula: $f_t(x, y) = y^2 + tx^3 + x^5$ $(t \in \mathbf{R})$. It is clear that $f_t^{-1}(0)$ is topologically constant, but there exist no real analytic family $\sigma_t(x, y)$ with $\sigma_0(x, y) = (x, y)$ and $\sigma_t(0, 0) = (0, 0)$ of local coordinates of $\mathbb{R}^2$ in which the Newton boundaries of $f_t(x, y)$ are stable.

This example claims that the topological equivalence (homeomorphism) among real analytic singularities is too weak in some sense.

Next we consider the blow analytic equivalence among real analytic singularities proposed by T. C. Kuo (see [2]). It is slightly weaker than bianalytic and much stronger than homeomorphism. He investigated real analytic families of real analytic functions which admit blow analytic trivializations (abbreviated to BAT) and he obtained beautiful results in his succesive papers (see [3]–[5]). We consider Oka's result for a real analytic family of real analytic functions which admits a BAT, and we obtain the following:

Theorem. *Let $f(x, y, t)$ be a real analytic family of real analytic functions of two variables x, y in a neighborhood of the origin, parametrized by $t \in I$, where I is an interval in $\mathbf{R}$ containing the origin. Assume that f admits a BAT along I and the Newton boundary of f_0 intersects the x, y-axes at two points. Then there exists a real analytic t-parameter family of local coordinates $\sigma_t(x, y) = (x(t), y(t))$ $(|t| \ll 1)$ such that*

(i) $\sigma_t(0, 0) = (0, 0)$ *and* $\sigma_0(x, y) = (x, y)$,

(ii) $\Gamma(f_t; \sigma_t) = \Gamma(f_0; (x, y))$,

where f_t is the restriction of f to $N \times \{t\}$ and $\Gamma(f_t; \sigma_t)$ denotes the Newton boundary of f_t associated with the coordinate $\sigma_t(x, y)$.

This result ensures that the blow analytic equivalence is strong enough and it is as fruitful as the topological equivalence in the complex case. It is also the converse of Theorems A and B in [1] in a wide sense.

In §1 we will explain some definitions of our terms and we will state our results more precisely. We will present their proofs in subsequent sections.

1. Results and definitions

Let $f(x, y)$ be a germ of a real analytic function defined in a neighborhood of $o \in \mathbb{R}^2$ and assume that $f(o) = 0$. Let $\sum a_{ij} x^i y^j$ be the Taylor series of f at the origin. Let $\Gamma_+(f; (x, y))$ be the convex hull of the union

$$\bigcup_{i,j} \{(i, j) + \mathbb{R}_+^2 | a_{ij} \neq 0\} \qquad (\mathbb{R}_+ = \{x \in \mathbb{R} | x \geq 0\}).$$

Let $\Gamma(f; (x, y))$ be the union of compact faces of $\Gamma_+(f; (x, y))$ and call $\Gamma_+(f; (x, y))$ the *Newton polygon* of f associated with the coordinate system

(x, y) and $\Gamma(f; (x, y))$ the *Newton boundary* of f associated with the coordinate system (x, y). We say that f is *convenient* if $\Gamma(f; (x, y))$ intersects the x, y-axes at two points. For any face $\gamma \in \Gamma(f; (x, y))$ we set

$$f_\gamma(x, y) = \sum_{(i, j) \in \gamma} a_{ij} x^i y^j$$

and we call $f_\gamma(x, y)$ the *restriction* of $f(x, y)$ to the face γ.

Let $\mathbb{R}^2(u, v)$ and $\mathbb{R}^2(u', v')$ be copies of $\mathbb{R}^2$, where (u, v) and (u', v') are coordinate systems of $\mathbb{R}^2$. Let $\mathscr{M}$ be the Möbius strip, that is, the 2-manifold which has two coordinate patches $\mathbb{R}^2(u, v)$ and $\mathbb{R}^2(u', v')$ with $uv = v'$ and $u'v = 1$. Note that

$$\mathbb{R}^2(u, v) \cap \mathbb{R}^2(u', v') = \{u' \neq 0\} = \{v \neq 0\}.$$

Let $\rho: \mathscr{M} \to \mathbb{R}^2$ be the map defined by $\rho(u, v) = (u, uv)$ and $\rho(u', v') = (u'v', v')$. Note that the restriction

$$\rho|_{\mathscr{M} - \rho^{-1}(o)}: \mathscr{M} - \rho^{-1}(o) \to \mathbb{R}^2 - \{o\}$$

is bianalytic. We call ρ *the blowing-up of* $\mathbb{R}^2$ with center the origin.

More generally we define the blowing-up of a manifold with center a point. Let $\mathscr{N}$ be a real analytic manifold of dim 2 withou. boundary and let U be a coordinate neighborhood of $p \in \mathscr{N}$. Let (x, y) be a coordinate system of U, i.e., $U \simeq \mathbb{R}^2(x, y)$. Let $\widetilde{\mathscr{N}}$ be the real analytic manifold obtained by attaching $\mathscr{N} - \{p\}$ to $\mathscr{M}$ by means of the map

$$\rho|_{\mathscr{M} - \rho^{-1}(o)}: \mathscr{M} - \rho^{-1}(o) \to \mathbb{R}^2 - \{0\} \simeq U - \{p\},$$

and let $\pi: \widetilde{\mathscr{N}} \to \mathscr{N}$ be the map such that the following diagram commutes:

$$\begin{array}{ccccc}
 & & \widetilde{\mathscr{N}} & \xleftarrow{\;i_2\;} & \mathscr{M} \\
 & \overset{i_1}{\nearrow} & \downarrow \pi & & \downarrow \rho \\
\mathscr{N} - \{p\} & \underset{i_1}{\searrow} & & & \\
 & & \widetilde{\mathscr{N}} & \xleftarrow{\;i_3\;} & U \simeq \mathbb{R}^2(x, y)
\end{array}$$

DIAGRAM 1.1

where i_k $(k = 1, 2, 3)$ are inclusions. Then it follows that $\widetilde{\mathscr{N}}$ and π are independent of a choice of coordinate systemes of U up to isomorphisms (see the construction of $\widetilde{\widetilde{H}}$ in §3). We call π the *blowing-up* of $\mathscr{N}$ with center $p \in \mathscr{N}$.

We recall the definition of BAT (MAT) in [2]. Let $I = [a, b]$ be a closed interval in $\mathbf{R}$, and $\widetilde{H}$ a homeomorphism between two neighborhoods $\widetilde{\mathscr{U}}$, $\widetilde{\mathscr{V}}$ of $\pi^{-1}(o) \times I$ in $\widetilde{\mathscr{N}} \times I$ with the following properties:

(1) $\widetilde{H}$ is t-level preserving;

(2) it is bianalytic;

(3) it leaves $\pi^{-1}(p) \times I$ invariant.

Then $\widetilde{H}$ induces the t-level preserving homeomorphism H between two neighborhoods $\mathscr{U}$, $\mathscr{V}$ $(\mathscr{U} = \pi(\widetilde{\mathscr{U}})$, $\mathscr{V} = \pi(\widetilde{\mathscr{V}}))$ of $\{p\} \times I$ in $\mathscr{N} \times I$ which leaves $\{p\} \times I$ pointwise fixed. We call such a homeomorphism a *blow-analytic twisting* of $\mathscr{N} \times I$ along I with center $\{p\} \times I$.

Suppose that I contains the origin, and let $f : \mathscr{N} \times I \to \mathbb{R}$ be a real analytic function with $f(p, t) = 0$. We say that f admits a *blow-analytic trivialization* (abbreviated to BAT) along I with center $\{p\} \times I$ if there exists a blow-analytic twisting H of $\mathscr{N} \times I$ along I with center $\{p\} \times I$ such that $f \circ H$ is independent of $t \in I$. Then replacing the U by a smaller neighborhood if necessary, we get

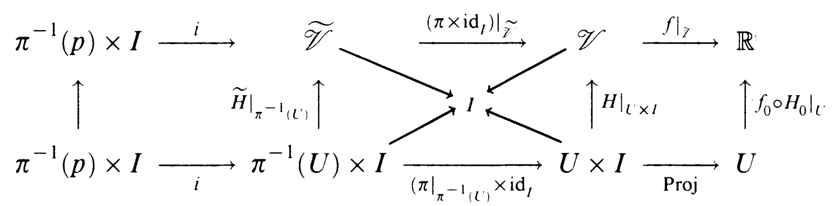

DIAGRAM 1.2

where $f_0 := f|_{\mathscr{N} \times \{0\}}$, $H_0 := H|_{\mathscr{U} \cap \mathscr{N} \times \{0\}}$, the four slopping arrows are projections on I and the i's are inclusions.

From now on we shall use the same notations for germs of functions at a point as their representations in a neighborhood of the point if no confusion should arise. Let (x, y) be a coordinate system of the neighborhood U of $p \in \mathscr{N}$, and Δ_t the set $\{(x, y) \in \mathbb{R}^2 | (x, y) \in \Gamma(f_t; (x, y))$, $x + y =$ the order of $f_t\}$. Now we have the following three lemmas.

Lemma 1. *Assume that f admits a BAT along I with center $\{p\} \times I$ and assume that*

$$f_{0, \Delta_0}(x, y) = x^\alpha y^\beta \left(\sum_{i=0}^{\gamma} a_i x^{\gamma - i} y^i \right), \qquad a_0 a_\gamma \neq 0,$$

as a germ at the origin. Then there exist germs $\varepsilon(t)$, $\delta(t)$ and $a_i(t)$ $(i = 1, \dots, \gamma)$, at $t = 0$, of real analytic functions which satisfy the following conditions;

(i) $f_{t, \Delta_t}(x, y) = (x - \delta(t) y)^\alpha (y - \varepsilon(t) x)^\beta (\sum_{i=0}^{\gamma} a_i(t) x^{\gamma - i} y^i)$,

(ii) $\varepsilon(0) = \delta(0) = 0$ *and* $a_i(0) = a_i$ $(i = 1, \dots, \gamma)$,

(iii) $\sum a_i(t) x^{\gamma - i} y^i$ *does not divide by* $(x - \delta(t) y)$ *or* $(y - \varepsilon(t) x)$ *in* $\mathbb{R}\{x, y\}$,

Lemma 2. *Assume that f admits a BAT along I with center $\{p\} \times I$ and assume that $\Delta_t = \Delta_0$ for any $t \in I$ and $f_{0, \Delta_0}(x, y)$, as a germ at the origin, has no powers of x (resp. y) only. Then there exists a positive number η such that the map*

$$f \circ (\pi \times \mathrm{id}_{I_\eta}) : \widetilde{\mathscr{N}} \times I_\eta \to \mathbb{R} \qquad (I_\eta = [-\eta, \eta])$$

admits a BAT longs I_η *with center* $\{o\} \times I_\eta$ *(resp.* $\{o'\} \times I_\eta$*), where* o *(resp.* o'*) is the origin of the coordinate patch* $\mathbb{R}^2(u, v)$ *(resp.* $\mathbb{R}^2(u', v')$*) of* $\pi^{-1}(U)$.

Lemma 3. *Assume that* f *admits a BAT along* I *with center* $\{p\}\times I$ *and assume that noncompact faces of* $\Gamma_+(f_t; (x, y))$ *are independent of* $t \in I$. *Then we have*

$$\Gamma(f_t : (x, y)) = \Gamma(f_0 : (x, y)) \quad \textit{for } |t| \ll 1.$$

Suppose that $\mathcal{N} = U \simeq \mathbb{R}^2(x, y)$, and we consider the real analytic function $f : U \times I \to \mathbb{R}$. Then we obtain our main theorem from these lemmas.

Theorem. *Assume that* f *admits a BAT along* I *with center* $\{p\} \times I$ *and* $f_0(x, y)$*, as a germ at the origin, is convenient. Then there exists a real analytic family of local coordinates* $\sigma_t(x, y) = (x(t), y(t))$ $(|t| \ll 1)$ *of* U *such that*

(i) $\sigma_t(0, 0) = (0, 0)$ *and* $\sigma_o(x, y) = (x, y)$,
(ii) $\Gamma(f_t; \sigma_t) = \Gamma(f_0; (x, y))$,

where f_t *is the restriction of* f *to* $\mathcal{N} \times \{t\}$ *and* $\Gamma(f_t; \sigma_t)$ *is the Newton boundary of the germ of* f_t *at the origin associated with the coordinate* $\sigma_t(x, y)$.

Remark. If $f_0(x, y)$ has an isolated singularity at the origin, the generality would not be lost under the hypothesis "convenience" since $f_0(x, y)$ is finitely determined, i.e., the analytic type of $f_0(x, y)$ is invariant after adding terms with higher degrees than some one.

We will prove the above-mentioned results in the following sections.

2. Proof of Lemma 1

We obtain the following addition to Diagram 1.2:

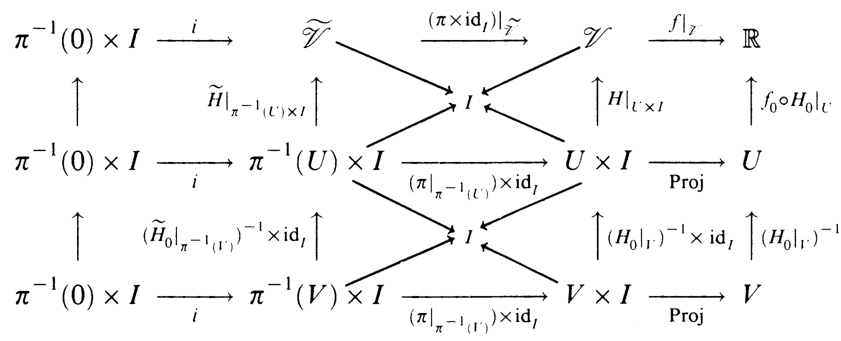

Diagram 2.1

where V is a small neighborhood of $p \in \mathcal{N}$, and $\widetilde{H}_0$ denotes the restriction of $\widetilde{H}$ to $\mathcal{U} \cap \widetilde{\mathcal{N}} \times \{0\}$. We set $\overline{H} = \widetilde{H} \circ ((\widetilde{H}_0|_{\pi^{-1}(V)})^{-1} \times \mathrm{id}_I)$. Then f admits a BAT via the blow-analytic twisting $H \circ (H_0|_V)^{-1} \times \mathrm{id}_I$ of $\mathcal{N} \times I$ with center $\{p\} \times I$ induced by $\overline{H}$ because $\overline{H}$ has the following properties:

(i) It is a t-level preserving bianalytic mapping.

(ii) It leaves $\pi^{-1}(o) \times I$ invariant.

(iii) $f \circ (\pi \times \mathrm{id}_I) \circ \overline{H}$ is independent of $t \in I$.

And moreover $\overline{H}$ has another property

(iv) $\overline{H}_0$ is the identity, where $\overline{H}_t$ denotes the restriction of $\overline{H}$ to $\pi^{-1}(V) \times \{t\}$.

First we work in the coordinate patch $\mathbb{R}^2(u, v)$ of $\pi^{-1}(U) \simeq \mathscr{M}$. There exists a neighborhood $\mathscr{W}$ of the origin in $\mathbb{R}^2(u, v) \times I$ such that $\overline{H}^{-1}(\mathscr{W}) \subset \mathbb{R}^2(u, v) \times I$ since $\overline{H}^{-1}$ has the origin as a fixed point. We set $(0, \varepsilon(t), t) := \overline{H}(0, 0, t) \in \mathscr{W}$ for $(0, 0, t) \in \overline{H}^{-1}(\mathscr{W})$. Then $\varepsilon(t)$ is a real analytic function with $\varepsilon(0) = 0$ because of the real analycity of $\overline{H}$ and $\overline{H}_0 = \mathrm{id}_{\pi^{-1}(V)}$. We set

$$\overline{H}^{-1}|_{\mathscr{W}}(u, v, t) =: (\varphi(u, v, t), \psi(u, v, t), t) \in \mathbb{R}^2(u, v) \times I,$$
$$\mathscr{W}_t := \mathscr{W} \cap (\mathbb{R}^2(u, v) \times \{t\}),$$
$$\varphi_t := \varphi|\mathscr{W}_t, \qquad \psi_t := \psi|\mathscr{W}_t.$$

Then by making $\mathscr{W}_t$ smaller if necessary, we can represent the real analytic functions φ_t, ψ_t in the neighborhood $\mathscr{W}_t$ of $(0, \varepsilon(t))$ as

$$\varphi_t(u, v) = u \cdot \varphi_t'(u, v),$$
$$\psi_t(u, v) = (v - \varepsilon(t)) \cdot \psi_t'(u, v) + u \cdot \psi_t''(u, v),$$

where $\varphi_t'(0, \varepsilon(t)) \neq 0$ and $\psi_t'(0, \varepsilon(t)) \neq 0$.

On the other hand we can write, in $\mathscr{W}_0$,

$$f_0 \circ \pi(u, v) = u^{\alpha+\beta+\gamma} \sum_{i=0}^{\gamma} a_i v^{\beta+i} + u^{\alpha+\beta+\gamma+1} f_0'(u, v).$$

Thus we have, in $\mathscr{W}_t$,

$$\begin{aligned} f_t \circ \pi(u, v) &= f_0 \circ \pi \circ \overline{H}_t^{-1}(u, v) \\ &= \varphi_t^{\alpha+\beta+\gamma}(u, v) \sum_{i=0}^{\gamma} a_i \psi_t^{\beta+i}(u, v) \\ &\quad + \varphi_t^{\alpha+\beta+\gamma+1}(u, v) f_0'(\varphi_t(u, v), \psi_t(u, v)) \\ &= u^{\alpha+\beta+\gamma} \sum_{i=0}^{\infty} a_i'(t)(v - \varepsilon(t))^{\beta+i} + u^{\alpha+\beta+\gamma+1} f_t'(u, v), \end{aligned}$$

where the $a_i'(t)$ $(i \geq 0)$ are real analytic in t. Since the first part in the last side is equal to $f_{t,\Delta_t} \circ \pi(u, v)$, it must be a polynomial of degree $\alpha + \beta + \gamma$ with respect to v and we have, in $\mathscr{W}_t$,

$$\begin{aligned} f_{t,\Delta_t} \circ \pi(u, v) &= u^{\alpha+\beta+\gamma} \sum_{i=0}^{\alpha+\gamma} a_i'(t)(v - \varepsilon(t))^{\beta+i} \\ &= (uv - \varepsilon(t)u)^{\beta} \sum_{i=0}^{\alpha+\gamma} a_i'(t) u^{\alpha+\gamma-i}(uv - \varepsilon(t)u)^i. \end{aligned}$$

Thus we obtain

$$(*) \qquad f_{t,\Delta_t}(x, y) = (y - \varepsilon(t)x)^{\beta} \sum_{i=0}^{\alpha+\gamma} a_i'(t)x^{\alpha+\gamma-i}(y - \varepsilon(t)x)^i$$

in an open set $\pi(\mathscr{W}_t - \{u = 0\}) \subset \mathscr{N}$ whose topological closure contains the origin of $\mathbb{R}^2(x, y)$. Hence it follows that $(*)$ holds in a neighborhood of the origin by the theorem of identity.

Since $a_0'(0) = a_0$, we obtain $a_0'(t) \neq 0$ for any t $(|t| \ll 1)$, and thus

$$(y - \varepsilon(t)x) \nmid \sum_{i=0}^{\alpha+\gamma} a_i'(t)x^{\alpha+\gamma-i}(y - \varepsilon(t)x)^i \quad \text{in } \mathbb{R}\{x, y\},$$

that is, $\sum_{i=0}^{\alpha+\gamma} a_i'(t)x^{\alpha+\gamma-i}(y - \varepsilon(t)x)^i$ does not divide by the factor $(y - \varepsilon(t)x)$ in the ring $\mathbb{R}\{x, y\}$.

Secondly by the same argument in another coordinate patch $\mathbb{R}^2(u', v')$, it follows that there exist real analytic functions $\delta(t)$, $a_i''(t)$ $(i = 0, \dots, \beta + \gamma)$ in t $(|t| \ll 1)$ with $\delta(0) = 0$ such that

$$f_{t,\Delta_t}(x, y) = (x - \delta(t)y)^{\alpha} \sum_{i=0}^{\gamma+\beta} a_i''(t)y^{\beta+\gamma-i}(x - \delta(t)y)^i$$

in a neighborhood of the origin and

$$(x - \delta(t)y) \nmid \sum_{i=0}^{\gamma+\beta} a_i''(t)y^{\beta+\gamma-i}(x - \delta(t)y)^i \quad \text{in } \mathbb{R}\{x, y\}.$$

Since the factors $(x-\delta(t)y)$ and $(y-\varepsilon(t)x)$ are relatively prime in $\mathbb{R}\{x, y\}$, we obtain

$$f_{t,\Delta_t}(x, y) = (x - \delta(t)y)^{\alpha}(y - \varepsilon(t)x)^{\beta} \sum_{i=0}^{\gamma} a_i(t)x^{\gamma-i}y^i$$

in a neighborhood of the origin of $\mathbb{R}^2(x, y)$ for any t $(|t| \ll 1)$. It is trivial that $a_i(t)$ $(i = 0, \dots, \gamma)$ are real analytic in t and $a_i(0) = a_i$, $(i = 0, \dots, \gamma)$. This completes the proof. □

3. Proof of Lemma 2

Assume that $f_{0,\Delta_0}(x, y) = x^{\alpha}y^{\beta}(\sum_{i=0}^{\gamma} a_i x^{\gamma-i}y^i)$, $a_0 a_\gamma \neq 0$. Then by means of Lemma 1 we can find real analytic functions $\delta(t)$, $\varepsilon(t)$ as in Lemma 1 so that

$$f_{t,\Delta_t}(x, y) = (x - \delta(t)y)^{\alpha}(y - \varepsilon(t)x)^{\beta} \left(\sum_{i=0}^{\gamma} a_i(t)x^{\gamma-i}y^i\right)$$

for any $t \in I_\eta = [-\eta, \eta]$ (n small enough). If $\varepsilon(t) \neq 0$ for some $t \in I_\eta$, the $f_{t,\Delta_t}(x, y)$ must contain the term $x^{\alpha+\beta+\gamma}$ with nonzero coefficient. But this contradicts our hypothesis, and we obtain $\varepsilon(t) = 0$ for any $t \in I_\eta$. Then the map $\overline{H}$ in the proof of Lemma 1 has an addition to the properties (i)–(iv): the property $\overline{H}(0, 0, t) = (0, 0, t)$ for any $t \in I_\eta$. Now let $\pi_1 : \widetilde{\widetilde{\mathscr{N}}} \to \widetilde{\mathscr{N}}$ be the blowing-up of $\widetilde{\mathscr{N}}$ with center $\{o\}$ (the o is the origin of $\mathbb{R}^2(u, v) \subset \widetilde{\mathscr{N}}$). Then there exists a bianalytic map $\widetilde{\overline{H}}$ such that the diagram

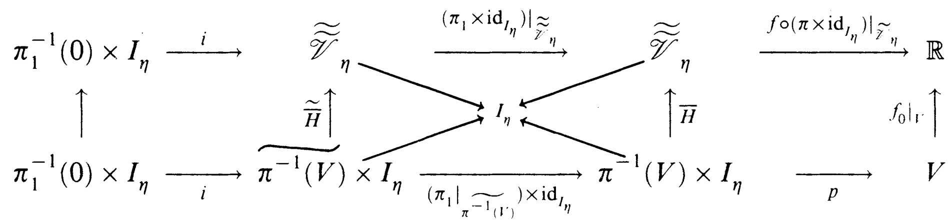

DIAGRAM 3.1

commutes, where $p := \operatorname{Proj} \circ (\pi|_{\pi^{-1}(V)} \times \mathrm{id}_{I_\eta})$, $\widetilde{\mathscr{V}}_\eta := V \cap (\mathscr{N} \times I_\eta)$, $\widetilde{\widetilde{\mathscr{V}}}_\eta := (\pi_1 \times \mathrm{id}_{I_\eta})^{-1}(\widetilde{\mathscr{V}}_\eta)$, $\widetilde{\pi^{-1}(V)} := \pi_1^{-1}(\pi^{-1}(V))$, and the four sloping arrows and the i's are as in Diagram 1.2. In fact the map $\widetilde{H}$ is constructed as follows. Let

$$\widetilde{\overline{H}}' : (\pi^{-1}(V) - \pi_1^{-1}(o)) \times I_\eta \to \widetilde{\widetilde{\mathscr{V}}}_\eta$$

be the lift of $\overline{H}$, that is, the injective bianalytic map defined by

$$\widetilde{\overline{H}}' = (\pi_1 \times \mathrm{id}_{I_\eta}|_{\widetilde{\mathscr{V}}_\eta - \{0\} \times I_\eta}) \circ \overline{H}|_{(\pi^{-1}(V) - \{o\}) \times I_\eta} \circ (\pi_1 \times \mathrm{id}_{I_\eta}|_{(\pi^{-1}(V) - \pi_1^{-1}(0)) \times I_\eta}).$$

Then it is clear that this map and its inverse can be extended analytically beyond $\pi^{-1}(o) \times I_\eta$, so that the extended map $\widetilde{\overline{H}}$ is bianalytic. Thus $\overline{H}$ is a blow-analytic twisting of $\widetilde{\mathscr{N}} \times I_\eta$ with center $\{o\} \times I_\eta$ induced by $\widetilde{\overline{H}}$ and it follows that $f \circ (\pi \times \mathrm{id}_{I_\eta})$ admits a BAT along I_η with center $\{o\} \times I_\eta$.

By the same argument in $\mathbb{R}^2(u', v')$ it follows that $f \circ (\pi \times \mathrm{id}_{I_\eta})$ admits a BAT along I_η with center $\{o'\} \times I_\eta$. This completes the proof. □

4. Proof of Lemma 3

We will prove this lemma by using the method analogous to Oka's in [7]. It is a well-known result that there exists a successive sequence $\pi_0, \ldots, \pi_k$ of blowing-ups with center a point such that $f_0 \circ \pi_0 \circ \cdots \circ \pi_k$ can be represented locally in terms of a product of powers of suitable local coordinates in a neighborhood of $(\pi_k \circ \cdots \circ \pi_0)^{-1}(p)$ (in other words we can resolve the singularity of f_0 at p).

Let $q(f_0; p)$ be the minimum number of succesive blowing-ups by which we can resolve the singularity of f_0 at p. We will prove this lemma by means of the induction on the number $q(f_0; p)$.

(1) *Case of* $q(f_0; p) = 0$. We can represent f_0 locally near p as follows:

$$f_0(x, y) = \varepsilon(x, y)(ax + by + \text{ higher})^\alpha (cx + dy + \text{ higher})^\beta,$$

where $a, \beta \in \mathbb{N} \cup \{0\}$, $ad - bc \neq 0$ and $\varepsilon(x, y)$ is a unit in the ring $\mathbb{R}\{x, y\}$.
First suppose that $ad \neq 0$. Then we can write

$$f_0(x, y) = \varepsilon(x, y)\{xh_1(x, y) + y^\lambda k_1(y)\}^\alpha \{yh_2(x, y) + x^\mu k_2(x)\}^\beta,$$

where h_i $(i = 1, 2)$ are units in $\mathbb{R}\{x, y\}$ and $k_1 \in \mathbb{R}\{y\}$ (resp. $k_2 \in \mathbb{R}\{x\}$) with $k_1(y) \equiv 0$ (resp. $k_2(x) \equiv 0$) or $k_1(0) \neq 0$ (resp. $k_2(0) \neq 0$). We consider the following four cases.

Case 1. *The case of* $k_1(0) \neq 0$ and $k_2(0) \neq 0$. Then $f_0(x, y)$ is clearly given by the formula

$$f_0(x, y) = \sum_{i=0}^{\alpha} \sum_{j=0}^{\beta} c_{ij}(x, y) x^{\alpha - i + \mu j} y^{\beta - j + \lambda i},$$

where $c_{ij}(x, y)$ $(i = 0, \dots, \alpha;\ j = 0, \dots, \beta)$ are units in $\mathbb{R}\{x, y\}$. We set $p = \alpha - i + \mu j$, $q = \beta - j + \lambda i$. Then we have

$$p + \mu q = \alpha + \mu\beta + i(\lambda\mu - 1) \geq \alpha + \mu\beta, \qquad q + \lambda p = \beta + \lambda\alpha + j(\lambda\mu - 1) \geq \beta + \lambda\alpha$$

and thus $\Gamma(f_0; (x, y))$ is illustrated in Figure 4.1.
If $\mu > 1$, then

$$f_{0, \Delta_0}(x, y) = y^\beta (c_0 x^\alpha + \cdots), \qquad c_0 \neq 0.$$

Since f admits a BAT along I with center $\{p\} \times I$, by means of Lemma 1 we obtain the formula:

$$f_{t, \Delta_t}(x, y) = (y - \varepsilon(t)x)^\beta (c(t), x^\alpha + \cdots) \quad \text{for any } t \ (|t| \ll 1),$$

where $c(t)$, $\varepsilon(t) \in \mathbb{R}\{t\}$ with $\varepsilon(0) = 0$ and $c(0) = c_0$. Suppose that $\varepsilon(t) \not\equiv 0$. Then $f_{t, \Delta_t}(x, y)$ contains the term $x^{\alpha+\beta}$ with nonzero coefficient, but this contradicts the hypothesis. Thus $\varepsilon(t) \equiv 0$.
Now we consider the function $f_0 \circ \pi$ in a neighborhood of the origin of the coordinate patch $\mathbb{R}^2(u, v)$ of $\widetilde{\mathcal{N}}$. We obtain the formula

$$f_0 \circ \pi(u, v) = f_0(u, uv) = \tilde{\varepsilon}(u, v) u^{\alpha+\beta} \{vh_2(u, uv) + u^{\mu-1} k_2(u)\}^\beta,$$

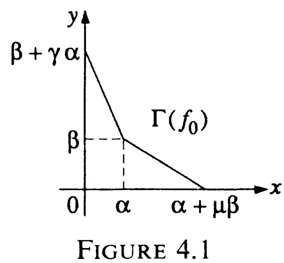

FIGURE 4.1

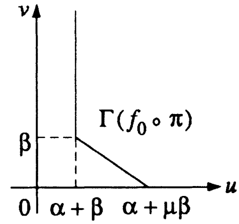

FIGURE 4.2

where $\tilde{\varepsilon}$ is a unit of $\mathbb{R}^2\{u, v\}$. Noting that $h_2(u, uv)$ is a unit of $\mathbb{R}^2\{u, v\}$, $\Gamma(f_0 \circ \pi; (u, v))$ is illustrated as follows:

Next we consider real analytic functions φ_t, ψ_t in $\mathscr{W}_t$ as in the proof of Lemma 1. Since $\varepsilon(t) \equiv 0$, we obtain

$$\varphi_t(u, v) = u\varphi_t'(u, v), \qquad \psi_t(u, v) = v\psi_t'(u, v) + u\psi_t''(u, v)$$

for any t ($|t| \ll 1$), where φ_t', ψ_t' are units in $\mathbb{R}\{u, v\}$. Thus

$$f_t \circ \pi(u, v) = f_0 \circ \pi \circ \overline{H}_t^{-1}(u, v) = \tilde{\tilde{\varepsilon}}(u, v)u^{\alpha+\beta}\{v\tilde{h}_2(u, v) + u^\rho \tilde{k}_2(u)\}^\beta,$$

where $\rho \in \mathbb{N} \cup \{0\}$, $\tilde{\tilde{\varepsilon}}$ and $\tilde{h}_2$ are units in $\mathbb{R}\{u, v\}$. By the hypothesis that noncompact faces of $\Gamma_+(f_t; (x, y))$ are stable, we obtain that $\rho = \mu - 1$ and $\tilde{k}_2$ is a unit in $\mathbb{R}\{u\}$. Thus

$$\Gamma(f_t \circ \pi; (u, v)) = \Gamma(f_0 \circ \pi; (u, v)) \quad \text{for any } t \ (|t| \ll 1).$$

Since $\Gamma(f_t \circ \pi; (u, v))$ maps to $\Gamma(f_t; (x, y))$ by the correspondence $\binom{u}{v} \to \binom{u-v}{v}$, we obtain

$$\Gamma(f_t; (x, y)) \cap \{(x, y) \in \mathbb{R}^2 | \alpha \le x\} = \Gamma(f_0; (x, y)) \cap \{(x, y) \in \mathbb{R}^2 | \alpha \le x\}$$

for any t ($|t| \ll 1$).

If $\mu = 1$, it is clear that the above formula holds.

On the other hand, in the same way as above we obtain

$$\Gamma(f_t; (x, y)) \cap \{(x, y) \in \mathbb{R}^2 | \beta \le y\} = \Gamma(f_0; (x, y)) \cap \{(x, y) \in \mathbb{R}^2 | \beta \le y\}$$

for any t ($|t| \ll 1$). Thus in Case 1

$$\Gamma(f_t; (x, y)) = \Gamma(f_0; (x, y)) \quad \text{for any } t \ (|t| \ll 1).$$

Case 2. *The case of* $k_1(y) \equiv 0$ and $k_2(0) \neq 0$.

Case 3. *The case of* $k_1(0) \neq 0$ and $k_2(y) \equiv 0$.

In each case $\Gamma(f_0; (x, y))$ is illustrated as in Figure 4.3 and Figure 4.4, corresponding to Case 2 and Case 3 respectively. In both cases we obtain the required result in the same way as Case 1.

Case 4. *The case of* $k_1(y) \equiv 0$ and $k_2(x) \equiv 0$. Then $\Gamma(f_0; (x, y))$ is illustrated as follows:

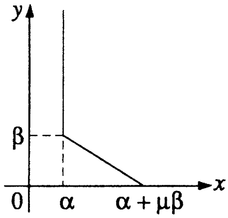

FIGURE 4.3

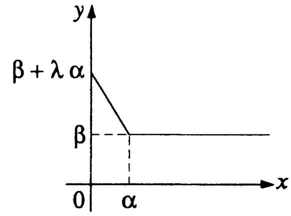

FIGURE 4.4

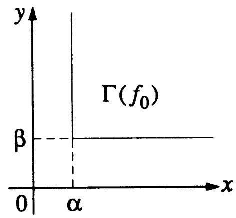

FIGURE 4.5

By the hypothesis we obtain the required result immediately in this case.

Secondly suppose that $ad = 0$. Then $bc \neq 0$ and we obtain the required result in the same way as in the case of $ad \neq 0$.

Hence the lemma is valid in the case of $q(f_0; p) = 0$.

(2) *Case of* $q(f_0; p) > 0$. We set

$$f_{0,\Delta_0}(x, y) := x^\alpha y^\beta \left(\sum_{i=0}^{\gamma} a_i x^{\gamma - i} y^i \right), \qquad a_0 a_\gamma \neq 0,$$

and there exist germs of real analytic function $\varepsilon(t)$, $\delta(t)$ and $a_i(t)$ as in Lemma 1 such that

$$f_{t,\Delta_t}(x, y) = (x - \delta(t)y)^\alpha (y - \varepsilon(t)x)^\beta \left(\sum_{i=0}^{\gamma} a_i(t) x^{\gamma - i} y^i \right)$$

for any t ($|t| \ll 1$). We consider the four cases according to the values of α and β.

In the case of $\alpha = \beta = 0$, both $(\gamma, 0)$ and $(0, \gamma)$ are in $\Gamma(f_{0,\Delta_0}; (x, y))$ and $\Gamma(f_{t,\Delta_t}; (x, y))$ for any t ($|t| \ll 1$) and thus we have

$$\Gamma(f_0; (x, y)) = \Gamma(f_{t,\Delta_t}; (x, y)) \quad \text{for any } t \ (|t| \ll 1).$$

In the case of $\alpha > 0$ and $\beta = 0$, if $\delta(t) \not\equiv 0$ then $(0, \alpha + \gamma) \in \Gamma(f_t; (x, y))$, but this contradicts the hypothesis. Thus we obtain $\delta(t) \equiv 0$.

In the case of $\alpha = 0$ and $\beta > 0$ (resp. $\alpha > 0$ and $\beta > 0$), if $\varepsilon(t) \not\equiv 0$ (resp. $\varepsilon(t) \not\equiv 0$ or $\delta(t) \not\equiv 0$) then we obtain the same contradiction as above. Thus we obtain $\varepsilon(t) \equiv 0$ (resp. $\varepsilon(t) \equiv 0$ and $\delta(t) \equiv 0$).

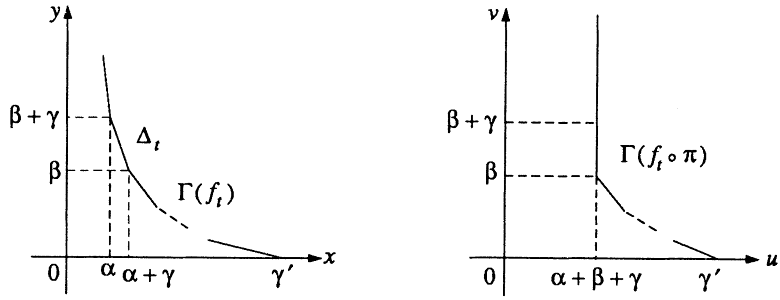

FIGURE 4.6

Hence in any case

$$f_{t,\Delta_t}(x, y) = x^{\alpha}y^{\beta}\left(\sum_{i=0}^{\gamma} a_i(t)x^{\gamma-i}y^i\right)$$

and

$$\Gamma(f_{t,\Delta_t}; (x, y)) = \Gamma(f_{0,\Delta_0}; (x, y))$$

for any t $(|t| \ll 1)$.

Suppose that $\beta > 0$. Note that $\Gamma(f_0 \circ \pi; (u, v))$ maps to $\Gamma(f_0; (x, y))$ by the correspondence $\binom{u}{v} \to \binom{u-v}{v}$. It is clear that one of the noncompact faces of $\Gamma(f_t \circ \pi; (u, v))$ is $\{(u, v)|u = \alpha+\beta+\gamma, \beta \leq v\}$ and it is independent of t $(|t| \ll 1)$. Another noncompact fact of it is invariant under the correspondence above. Thus noncompact faces of $\Gamma(f_t \circ \pi; (u, v))$ are stable for any t $(|t| \ll 1)$, and moreover, from Lemma 2 it follows that $f \circ (\pi \times \mathrm{id}_{I_\eta})$ admits a BAT along I_η with center $\{o\} \times I_\eta$ (the o is the origin of $\mathbb{R}^2(u, v)$). Since $q(f_0 \circ \pi; o) < q(f_0; p)$, from the hypothesis of our induction we obtain

$$\Gamma(f_0 \circ \pi; (u, v)) = \Gamma(f_t \circ \pi; (u, v)) \quad \text{for any } t \ (|t| \ll 1),$$

and thus

$$\begin{aligned}\Gamma(f_t; (x, y)) &\cap \{(x, y)|\alpha \leq x\} \\ &= \Gamma(f_0; (x, y)) \cap \{(x, y)|\alpha \leq x\} \quad \text{for any } t \ (|t| \ll 1).\end{aligned}$$

Suppose that $\alpha > 0$. Then we obtain

$$\begin{aligned}\Gamma(f_t; (x, y)) &\cap \{(x, y)|\beta \leq y\} \\ &= \Gamma(f_0; (x, y)) \cap \{(x, y))|\beta \leq y\} \text{ for any } t \ (|t| \ll 1)\end{aligned}$$

in the same way as in the case of $\beta > 0$.

Hence we obtain

$$\Gamma(f_t; (x, y)) = \Gamma(f_0; (x, y)) \quad \text{for any } t \ (|t| \ll 1).$$

This completes the proof of this lemma. □

5. Proof of Theorem

We use the method of [7]. Let (x', y') be a coordinate system for U. We set $U^{0\pm} = U$, $(u_{0\pm}, v_{0\pm}) = (u'_{0\pm}, v'_{0\pm}) = (x', y')$ and $o_{0\pm} = o'_{0\pm} = p \in U$.

Next we define inductively real analytic manifolds $U^{k\pm}$, real analytic maps $\rho^{k\pm}: U^{k\pm} \to U^{(k-1)\pm}$, points $o_{k\pm} \in U^{k\pm}$ and coordinates $(u_{k\pm}, v_{k\pm})$, $(u'_{k\pm}, v'_{k\pm})$ according to the signs $+, -$ respectively as follows. Let ρ^{k+}: $U^{k+} \to U^{(k-1)+}$ (resp. $\rho^{k-}: U^{k\pm} \to U^{(k-1)-}$) be the blowing-up of $U^{(k-1)+}$ (resp. $U^{(k-1)-}$) with center $o_{(k-1)+}$ (resp. $o'_{(k-1)-}$) and let $\mathbb{R}^2(u_{k+}, v_{k+})$ and $\mathbb{R}^2(u'_{k+}, v'_{k+})$ (resp. $\mathbb{R}^2(u_{k-}, v_{k-})$ and $\mathbb{R}^2(u'_{k-}, v'_{k-})$) be two coordinate patches of $(\rho^{k+})^{-1}(\mathbb{R}^2(u_{(k-1)+}, v_{(k-1)+}))$ (resp. $(\rho^{k-})^{-1}(\mathbb{R}^2(u'_{(k-1)-}, v'_{(k-1)-})))$ with $u_{k+}v_{k+} = v'_{k+}$ and $u'_{k+}v_{k+} = 1$ (resp. $u_{k-}v_{k-} = v'_{k-}$ and $u'_{k-}v_{k-} = 1$). Then

$$\mathbb{R}^2(u_{k\pm}, v_{k\pm}) \cap \mathbb{R}^2(u'_{k\pm}, v'_{k\pm}) = \{u'_{k\pm} \neq 0\} = \{v_{k\pm} \neq 0\},$$

$$\begin{aligned}\rho^{k+}(u_{k+}, v_{k+}) &= (u_{k+}, u_{k+}v_{k+}) = (u_{(k-1)+}, v_{(k-1)+}) \\ &= (u'_{k+}v'_{k+}, v'_{k+}) = \rho^{k+}(u'_{k+}, v'_{k+})\end{aligned}$$

and

$$\begin{aligned}\rho^{k-}(u_{k-}, v_{k-}) &= (u_{k-}, u_{k-}v_{k-}) = (u'_{(k-1)-}, v'_{(k-1)-}) \\ &= (u'_{k-}v'_{k-}, v'_{k-}) = \rho^{k-}(u'_{k-}, v'_{k-}).\end{aligned}$$

We set $\pi^{k\pm} = \rho^{1\pm} \circ \cdots \circ \rho^{k\pm}: U^{k\pm} \to U$. Note that

$$\pi^{k+}(u_{k+}, v_{k+}) = (u_{k+}, u_{k+}^k v_{k+}) \quad \text{and} \quad \pi^{k-}(u'_{k-}, v'_{k-}) = (u'_{k-}(v'_{k-})^k, v'_{k-}).$$

Now we set

$$\begin{aligned}&m_{0+} = \text{ the order of } f_0(x, y), \\ &m_{k+} = \text{ the order of } f_0 \circ \pi^{k+}(u_{k+}, v_{k+}) \quad (k \geq 1), \\ &\Delta'_{k+} = \{(u_{k+}, v_{k+}) \in \Gamma(f_0 \circ \pi^{k+}; (u_{k+}, v_{k+})) | u_{k+} + v_{k+} = m_{k+}\} \quad (k \geq 1).\end{aligned}$$

Let Δ_{0+} be the set $\{(x, y) \in \Gamma(f_0; (x, y)) | x + y = m_{0+}\}$ and let Δ_{k+} be the fact of $\Gamma(f_0; (x, y))$ corresponding to Δ'_{k+} by the map

$$\begin{pmatrix} u_{k+} \\ v_{k+} \end{pmatrix} \to \begin{pmatrix} u_{k+} - kv_{k+} \\ v_{k+} \end{pmatrix}.$$

Suppose that

$$f_{0,\Delta_0}(x, y) = x^{\alpha_1} y^{\beta_1} \left(\sum_{i=0}^{\gamma_1} a_i x^{\gamma_1 - i} y^i \right), \qquad a_0 a_\gamma \neq 0.$$

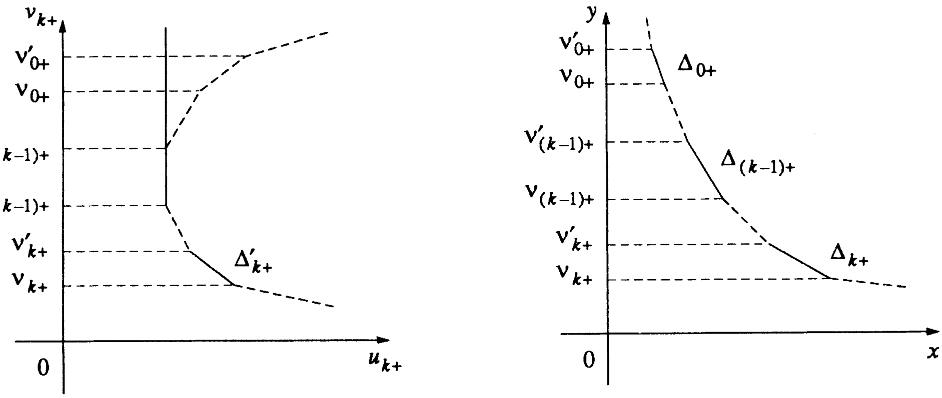

FIGURE 5.1

By Lemma 1 we can find germs of real analytic functions $\varepsilon_1(t)$, $\delta_1(t)$ and $a_i(t)$ $(i = 1, \dots, \gamma_1)$ as in Lemma 1 so that

$$f_{t,\Delta_t}(x, y) = (x - \delta_1(t)y)^{\alpha_1}(y - \varepsilon_1(t)x)^{\beta_1}\left(\sum_{i=0}^{\gamma_1} a_i(t)x^{\gamma_1 - i}y^i\right).$$

We set

$$\sigma^{1+}(x, y, t) = (x_1(t), y_1(t), t) = (x - \delta_1(t)y, y - \varepsilon_1(t)x, t).$$

Then $(x_1(t), y_1(t))$ is a real analytic family of coordinates of U and

$$\{(x_1, y_1) \in \Gamma(f_t \circ (\sigma_t^{1+})^{-1}; (x_1, y_1)) | x_1 + y_1 = \text{the order of } f_t \circ (\sigma_t^{1+})^{-1}(x_1, y_1)\} = \Delta_{0+}$$

for any t $(|t| \ll 1)$, where $\sigma_t^{1+} = \sigma^{1+}|_{\mathbb{R}^2(x_1, y_1) \times \{t\}}$.

Next we set

$$\nu_{k+} = \min\{y |\ \text{there exists } x \text{ such that } (x, y) \in \Delta_{k+}\},$$
$$\nu'_{k+} = \max\{y |\ \text{there exists } x \text{ such that } (x, y) \in \Delta_{k+}\}$$

and there exists $\gamma \in \mathbb{N}$ such that

$$\nu_{0+} \geq \cdots \geq \nu_{(\gamma-1)+} \geq \nu_{\gamma+} = 0, \qquad \nu'_{0+} \geq \cdots \geq \nu'_{(\gamma-1)+}.$$

We prove the following assertion by means of induction.

Assertion 5.1. *There exist a positive number η_{k+} and a bianalytic map*

$$\begin{aligned} \sigma^{k+} : \mathbb{R}^2(x, y) \times I_{\eta_{k+}} &\to \mathbb{R}^2(x_{k+}, y_{k+}) \times I_{\eta_{k+}} \qquad (k = 1, \dots, \gamma) \\ (x, y, t) &\to (x_{k+}(t), y_{k+}(t), t) \end{aligned}$$

such that

(1) $\sigma^{k+}(x, y, 0) = (x, y, 0)$, $\sigma^{k+}(0, 0, t) = (0, 0, t)$,
(2) $\Gamma(f_t \circ (\sigma_t^{k+})^{-1} ; (x_{k+}, y_{k+})) \cap \{(x_{k+}, y_{k+}) | \nu_{k+} \le y_{k+} \le \nu'_{k+}\} = \Delta_{k+}$,
(3) $\Gamma(f_t \circ (\sigma_t^{k+})^{-1} ; (x_{k+}, y_{k+})) \cap \{(x_{k+}, y_{k+}) | \nu_{1+} \le y_{k+}\}$
$= \Gamma(f_t \circ (\sigma_t^{1+})^{-1} ; (x_{1+}, y_{1+})) \cap \{(x_{1+}, y_{1+}) | \nu_{1+} \le y_{1+}\}$,

where $\sigma_t^{k+} = \sigma^{k+}|_{\mathbb{R}^2(x_{k+}, y_{k+}) \times \{t\}}$.

Proof of the assertion. There exist a positive number η_{k+} and a bianalytic map $\Sigma^{k+}: \widetilde{U} \times I_{\eta_{k+}} \to \widetilde{U} \times I_{\eta_{k+}}$ such that the diagram

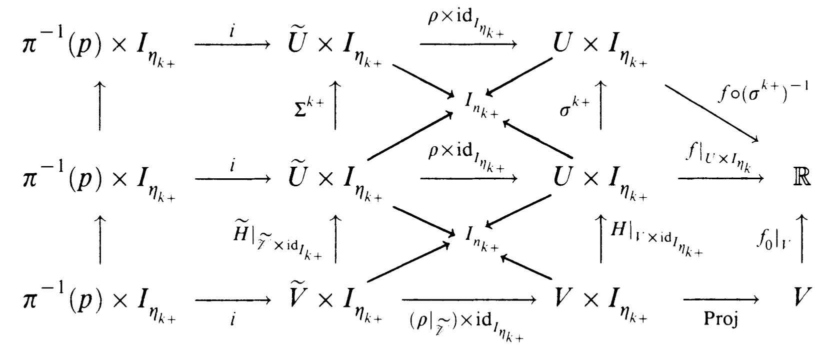

DIAGRAM 5.1

commutes, where V is a small neighborhood of $p \in U$ and $\widetilde{U} := \rho^{-1}(U)$, $\widetilde{V} := \rho^{-1}(V)$ and the four sloping arrows, the i's, $\widetilde{H}$, and H together are as in §1. Note that the Σ^{k+} can be constructed as $\widetilde{H}$ in §3. Thus $\sigma^{k+} \circ \overline{H}|_{V \times id_{I_{\eta_{k+}}}}$ is a blow-analytic twisting of $U \times I_{\eta_{k+}}$ with center $\{p\} \times I_{\eta_{k+}}$ and $f \circ (\sigma^{k+})^{-1}$ admits a BAT along $I_{\eta_{k+}}$ with center $\{p\} \times I_{\eta_{k+}}$.

Next we set $(x_{k+}, y_{k+}) = (x', y')$ and we consider the map $\pi^{k+}: U^{k+} \to U$. Recall that $\pi^{k+}(u_{k+}, v_{k+}) = (u_{k+}, u_{k+}^k v_{k+}) = (x_{k+}, y_{k+})$. By the hypothesis of the induction, the faces associated with the initial part of $f_t \circ (\sigma_t^{k+})^{-1} \circ \pi^{k+}(u_{k+}, v_{k+})$ are Δ'_{k+} and they are independent of $t \in I_{\eta_{k+}}$. Thus if $k < \gamma$, by Lemma 2 $f \circ (\sigma^{k+})^{-1} \circ (\pi^{(k+1)+} \times \mathrm{id}_{I_{(k+1)+}})$ admits a BAT along $I_{\eta_{(k+1)+}}$ with center $\{o_{(k+1)+}\} \times I_{\eta_{(k+1)+}}$ for some positive number $\eta_{(k+1)+}$. Suppose that

$$\text{the initial part of } f_0 \circ (\sigma_0^{k+})^{-1} \circ \pi^{(k+1)+}(u_{k+}, v_{k+})$$
$$= u_{k+}^{\alpha_k} v_{k+}^{\beta_k} \left(\sum_{i=0}^{\gamma_k} a_i^k u_{k+}^{\gamma_k - i} \cdot v_{k+}^i \right), \qquad a_0^k a_{\gamma_k}^k \ne 0.$$

Then by Lemma 1 we can find $\varepsilon_k(t)$ and $a_i^k(t)$ $(i = 1, \ldots, \gamma_k)$ such as $\varepsilon(t)$,

$a_i(t)$ $(i = 1, \dots, \gamma)$ in Lemma 1 so that

$$\begin{aligned}\text{the initial part of } & f_t \circ (\sigma_t^{k+})^{-1} \circ \pi^{(k+1)+}(u_{k+}, v_{k+}) \\ &= u_{k+}^{\alpha'_k}(v_{k+} - \delta(t)u_{k+})^{\beta_k}\left(\sum_{i=0}^{\gamma_k} a_i^k(t)u_{k+}^{\gamma_k - i} \cdot v_{k+}^i\right),\end{aligned}$$

where $\alpha'_k = \alpha_k - k(\beta_k + \gamma_k) = m_0 + \cdots + m_k - k(\beta_k + \gamma_k)$.

Now we set

$$\begin{aligned}\sigma^{(k+1)+}(x_{k+}, y_{k+}, t) &= (x_{(k+1)+}(t), y_{(k+1)+}(t), t) \\ &= (x_{k+1}, y_{k+1} - \varepsilon_k(t)x_{k+}^{k+1}, t).\end{aligned}$$

Then it follows that $\sigma^{(k+1)+}$ satisfies conditions (1), (2) and (3) in the assertion. It is trivial that (1) and (2) are satisfied by $\sigma^{(k+1)+}$. We can show that condition (3) is satisfied by $\sigma^{(k+1)+}$ as follows. We have

$$(*)\quad\begin{aligned}x_{k+}^p y_{k+}^q &= x_{(k+1)+}^p (y_{(k+1)+} - \varepsilon_k(t)x_{(k+1)+}^{k+1})^q \\ &= x_{(k+1)+}^p y_{(k+1)+}^q + \sum_{i=0}^{p} \binom{p}{i}(-\varepsilon_k(t))^i x_{(k+1)+}^{p+(k+1)i} y_{(k+1)+}^{q-i} \quad \text{for } k \ge 2.\end{aligned}$$

The Newton boundary of the last side of $(*)$ in the $(x_{(k+1)+}, y_{(k+1)+})$-plane is in the half-line emanating from the startpoint (p, q) running parallel to the face $\Delta_{(k+1)+}$ for $k \ge 2$. Thus it follows easily that if $(p, q) \in \Gamma_+(f_t \circ (\sigma_t^{k+})^{-1}; (x_{k+}, y_{k+}))$ then all the terms in the last side of $(*)$ are on or over the line containing each face

$$\gamma \in \Gamma(f_t \circ (\sigma_t^{(k+1)+})^{-1}; (x_{(k+1)+}, y_{(k+1)+})) \cap \{\nu_{k+1} \le y_{(k+1)+}\} \quad \text{for } k \ge 2.$$

Thus we have

$$\begin{aligned}\Gamma(f_t \circ (\sigma_t^{(k+1)+})^{-1}; (x_{k+1}, y_{k+1})) &\cap \{(x_{k+1} y_{k+1}) | \nu_{1+} \le y_{k+1}\} \\ = \Gamma(f_t \circ (\sigma_t^{1+})^{-1}; (x_{1+}, y_{1+})) &\cap \{(x_{1+}, y_{1+}) | \nu_{1+} \le y_{1+}\} \quad \text{for } k \ge 2.\end{aligned}$$

Hence condition (3) is satisfied by $\sigma^{(k+1)+}$. By the induction the assertion is proved.

Next we set $f' = f \circ (\sigma^{\gamma+})^{-1}$ and it follows that f' admits a BAT along $I_{\eta_{\gamma+}}$ with center $\{p\} \times I_{\eta_{\gamma+}}$. We do the same argument for $\Gamma(f'_0; (x_{\gamma+}, y_{\gamma+})) \cap \{(x_{\gamma+}, y_{\gamma+}) | \nu_{1+} \le y_{\gamma+}\}$ by using π^{k-} etc. We set

$$\begin{aligned} m_{0-} &= \text{ the order of } f'_0(x_{\gamma+}, y_{\gamma+}), \\ m_{k-} &= \text{ the order of } f'_0 \circ \pi^{k-}(u'_{k-}, v'_{k-}) \quad (k \ge 1), \\ \Delta'_{k-} &= \{(u'_{k-}, v'_{k-}) \in \Gamma(f'_0 \circ \pi^{k-}; (u'_{k-}, v'_{k-})) | u'_{k-} + v'_{k-} = m_{k-}\} \quad (k \ge 1).\end{aligned}$$

Let $\Delta_{0-} = \Delta_{0+}$ and let Δ_{k-} be the face of $\Gamma(f'_0; (x_{\gamma+}, y_{\gamma+}))$ corresponding to Δ'_{k-} by the map

$$\begin{pmatrix} u'_{k-} \\ v'_{k-} \end{pmatrix} \to \begin{pmatrix} u'_{k-} \\ v'_{k-} - ku'_{k-} \end{pmatrix}.$$

We set

$$\nu_{k-} = \min\{x \mid \text{there exists } y \text{ such that } (x, y) \in \Delta_{k-}\},$$
$$\nu'_{k-} = \max\{x \mid \text{there exists } y \text{ such that } (x, y) \in \Delta_{k-}\}$$

and there exists $\gamma' \in \mathbb{N}$ such that

$$\nu_{0-} \geq \cdots \geq \nu_{(\gamma'-1)-} \geq \nu_{\gamma'-} = 0, \qquad \nu'_{0-} \geq \cdots \geq \nu'_{(\gamma'-1)-}.$$

Then we obtain the following assertion in the same way as the previous assertion.

Assertion 5.2. *There exist a positve number η_{k-} and a bianalytic map*

$$\sigma^{k-}: \mathbb{R}^2(x_{\gamma+}, y_{\gamma+}) \times I_{\eta_{\gamma'-}} \to \mathbb{R}^2(x_{\gamma'-}, y_{\gamma'-}) \times I_{\eta_{\gamma'-}} \qquad (k = 1, \dots, \gamma),$$
$$(x_{\gamma'+}, y_{\gamma'+}, t) \to (x_{\gamma'-}(t), y_{\gamma'-}(t), t)$$

such that

(1) $\sigma^{k-}(x, y, 0) = (x, y, 0)$, $\sigma^{k-}(0, 0, t) = (0, 0, t)$,

(2) $\Gamma(f'_t \circ (\sigma_t^{k-})^{-1}; (x_{k-}, y_{k-})) \cap \{x_{k-}, y_{k-}) | \nu_{k-} \leq y_{k-} \leq \nu'_{k-}\} = \Delta_{k-}$,

(3) $\Gamma(f'_t \circ (\sigma_t^{k-})^{-1}; (x_{k-}, y_{k-})) \cap \{(x_{k-}, y_{k-}) | \nu_{1-} \leq y_{k-}\}$
$= \Gamma(f'_0; (x_{\gamma+}, y_{\gamma+})) \cap \{(x_{\gamma+}, y_{\gamma+}) | \nu_{1+} \leq y_{\gamma+}\}$,

where $\sigma_t^{k-} = \sigma^{k-}|_{\mathbb{R}^2(x_{\gamma+}, y_{\gamma+}, \times\{t\}}$.

Hence $\Gamma_+(f'_t \circ (\sigma_t^{k-})^{-1}; (x_{\gamma'-}, y_{\gamma'-}))$ has two invariant noncompact faces

$$\Gamma_+(f_0; (x, y)) \cap \{(x, y) | x = 0\} \quad \text{and} \quad \Gamma_+(f_0; (x, y)) \cap \{(x, y) | y = 0\}.$$

Since $f' \circ (\sigma^{\gamma'-})^{-1}$ admits a BAT along $I_{\eta_{\gamma'-}}$ with center $\{p\} \times I_{\eta_{\gamma'-}}$ as we showed in the proof of Assertion 5.1, by Lemma 3 we obtain

$$\Gamma(f'_t \circ (\sigma_t^{\gamma'-})^{-1}; (x_{\gamma'-}, y_{\gamma'-})) = \Gamma(f'_0 \circ (\sigma_0^{\gamma'-})^{-1}; (x_{\gamma'-}, y_{\gamma'-})).$$

Thus it is clear that $\sigma(x, y, t) = \sigma^{\gamma'} \circ \sigma^{\gamma+}(x, y, t)$ satisfes the conditions in Theorem. This completes the proof. □

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